Cut Sparsification

Definition: A $(1 \pm \varepsilon)$-cut sparsifier $H=(V, E_H)$ is a weighted subgraph of a given graph $G=(V, E_G)$ such that for all subsets $\emptyset \subset S \subset V$:

$$(1-\varepsilon)\,|E_G(S, V\setminus S)| \leq |E_H(S, V\setminus S)| \leq (1+\varepsilon)\,|E_G(S, V\setminus S)|$$

$$|E_H(S, V\setminus S)| = \sum_{\substack{(u,v)\in E_H \\ u\in S,\, v\in V\setminus S}} w_H(u,v)$$

Today: G undirected, unweighted

Straightforward idea:

→ Include each edge (u,v) of G in H with probability p (and if so, set $w_H(u,v) = \frac{1}{p}$) (in general $w_H(u,v) = \frac{1}{p} \cdot w_G(u,v)$)

Analysis

- Consider for every edge $(u,v) \in E_G$ random variable $X_{(u,v)} = \begin{cases} 1 & \text{if } (u,v) \text{ is sampled} \\ 0 & \text{otherwise} \end{cases}$

• Let $(S, V\setminus S)$ be an arbitrary cut

$$E[|E_H(S, V\setminus S)|] = E\Big[\sum_{\substack{(u,v)\in E_G \\ u\in S, v\in V\setminus S}} X_{(u,v)} \cdot \frac{1}{p}\Big] = \frac{1}{p} \sum_{\substack{(u,v)\in E_G \\ u\in S, v\in V\setminus S}} \overbrace{E[X_{u,v}]}^{p} = \frac{1}{p} \cdot p \cdot |E_G(S,V\setminus S)| = |E_G(S, V\setminus S)|$$

<u>But:</u> This only gives expectation for each cut

<u>Markov Bound:</u> $Pr[|E_H(S,V\setminus S)| > (1+\varepsilon)|E_G(S,V\setminus S)|] < \frac{1}{1+\varepsilon}$

$\hookrightarrow$ only constant probability, cannot use union bound to get guarantee for <u>all</u> (possibly exponentially many) cuts

$\hookrightarrow$ only gives a bound in one direction

<u>Chernoff Bound:</u>

$$Pr\Big[\sum_{(u,v)\in E_G} X_{(u,v)} > (1+\varepsilon)\cdot \underbrace{p\cdot |E_G(S,V\setminus S)|}_{\mu} \vee \sum_{(u,v)\in E_G} X_{(u,v)} < (1-\varepsilon)\cdot p\, E_G(S,V\setminus S)\Big]$$

$$\leq 2 \cdot \frac{1}{e^{\varepsilon^2/3 \cdot \mu}} \leq \frac{1}{n^c} \quad \text{if } \mu \geq \frac{3}{\varepsilon^2}\cdot (c+1)\cdot \ln(n)$$

$$\Rightarrow |E_G(S,V\setminus S)| \geq \frac{1}{p}\cdot \frac{3}{\varepsilon^2}(c+1)\ln(n)$$

$$2 \cdot \frac{1}{e^{\frac{\varepsilon^2}{3} \cdot \mu}} \leq 2 \cdot \frac{1}{e^{\cancel{\frac{\varepsilon^2}{3}} \cdot \cancel{\frac{3}{\varepsilon^2}} (c+1)\ln(n)}}$$

$n \geq 2$ because $n=1$ does not need spars.

$$= 2 \cdot \frac{1}{\left(e^{\ln(n)}\right)^{c+1}} = 2 \frac{1}{n^{c+1}} \leq \frac{n}{n^{c+1}} = \frac{1}{n^c}$$

<u>Summary</u>: By Chernoff Bound, each cut of size $\Omega\left(\frac{1}{p} \cdot \frac{\log(n)}{\varepsilon^2}\right)$ will be preserved with quality $(1 \pm \varepsilon)$ with high probability

<u>Size of H:</u>

$$E[|E_H|] = p \cdot |E_G| = p \cdot m \leq p \cdot n^2$$